On T-folds, G-structures and supersymmetry

James Gray

Centre for Particle Theory, Department of Mathematical Sciences
University of Durham
Durham DH1 3LE, U.K.
E-mail: `j.a.gray2@durham.ac.uk`

Emily Hackett-Jones

School of Mathematics, University of Edinburgh
Edinburgh EH9 3JZ, U.K.
E-mail: `E.Hackett-Jones@ed.ac.uk`

ABSTRACT: We describe how to calculate the amount of supersymmetry associated to a class of supergravity theories obtained by compactification on T-folds. We illustrate our discussion by calculating the degree of supersymmetry enjoyed by a particular set of massive supergravities which have been obtained in the literature by compactifying type II supergravity on such backgrounds. Our discussion involves a modification of the usual arguments, based upon G-structures, for the amount of supersymmetry preserved by geometric compactifications.

Contents

1. Introduction

The usual procedure for constructing background field configurations of string and M theory involves two steps. Firstly, the low energy fields are defined on coordinate patches in an atlas covering some manifold. A consistent global picture is then created in the second step by insisting that where coordinate patches overlap the different field configurations describe the same physical setup. This is usually achieved by relating the relevant local fields on different coordinate patches by diffeomorphisms and gauge transformations. We shall refer to such backgrounds as "geometric" as the resulting configurations include a globally well defined metric tensor.

One can carry out the second step of requiring a consistent global picture in other ways (see [1] for the first general consideration and discussion of this). Any symmetry of the theory can be used to join together coordinate patches while still fulfilling the criterion that the various field configurations on overlapping patches describe the same physics. For example, in a toroidally compactified type II string theory one could use elements of the $O(d,d)$ T-duality group as transition functions [1–16]. This results in what Hull has called a "T-fold" [1]. In general, there is no globally defined metric tensor on a T-fold. This is a consequence of the fact that T-duality transformations mix up the internal components of metric with those of the NS-NS two form. We will therefore follow the literature in referring to such spaces where the metric "jumps" between coordinate patches in this manner as "non-geometric". The idea of patching a manifold together with duality transformations,

to obtain these so-called "duality-folds" [1], has been pursued by a number of authors over the last few years (see [1 − 12] for this and related work). In passing we note that it appears that the use of supersymmetry in performing the 'second step' above has not yet been considered in the literature.

Given their unusual nature it is helpful to briefly review some of the evidence that string theory can make sense on such backgrounds. Firstly some T-folds can be shown to be T-dual to various, more standard, geometric spaces. For example, the T-dual of certain torus compactifications with NS-NS flux (with an appropriate domain wall solution in the external space) are T-folds (see for example [5, 7]). Thus one is left with two possibilities. Either the T-fold obtained in such a manner is as good a string background as the torus compactification with flux, or T-duality is not a feature of the full theory in that, for some reason, the torus compactification's T-dual is not a good string background.

A second piece of evidence which indicates that we should take T-fold backgrounds seriously within the context of string theory is provided by their relation to certain well defined conformal field theories (CFTs) [5]. It is well known that certain smooth geometric backgrounds of string theory have limits in which they become orbifolds of flat space. The associated world sheet CFT can then be described in detail. In a similar manner there are limits of certain T-fold compactifications which are associated with well understood consistent CFTs [5]. In this case the relevant orbifolds are *asymmetric* examples in which the left and right moving degrees of freedom on the string world sheet are acted on differently by the orbifolding [17].

Once it is accepted that it makes sense to consider string theory on these T-fold backgrounds the next obvious question is why are these configurations interesting? There are a multitude of interesting consequences which follow from considering string and M-theory on various symmetry folds. One strong motivation for studying such backgrounds is that they have been found to typically possess fewer unstabilised moduli than more conventional compactifications [5, 6, 8, 9]. It is reasonably straight forward to see why this is so. Unstabilised moduli in lower dimensional theories correspond to unspecified integration constants in the associated higher dimensional vacuum solutions. The use of a larger set of symmetries (such as the full T-duality group instead of just some "geometric" subgroup) in patching together a compactification manifold effectively constitutes a more restrictive set of boundary conditions on the possible vacuum solutions. As such the solutions tend to have fewer unspecified integration constants, leading to fewer unstabilised moduli in the lower dimensional theories. However, within the context of phenomenology, where considerations of moduli stabilisation would be most important, the use of these spaces raises various problems. This is because, for all values of the moduli, the known examples of non-trivial T-folds contain cycles of string scale size. This leads to concerns about the lack of a mass gap above the standard supergravity modes as well as concern that non-perturbative effects could well be important. These difficulties may well be resolvable in some cases. For example, it is easy to build examples of T-folds where one can show that, despite these comments, the size of non-perturbative effects in the superpotential is controlled by the volume of the base manifold and not that of the fibre [18]. In addition, there are such examples where only a finite number of extra light states would have to be

included before a mass gap with the rest of the spectrum is obtained. However, in the interests of being conservative, and to show that our work is of interest whether or not these issues are resolvable, we shall state our motivation in this paper in terms of a different use for T-fold compactification.

An uncontroversial use of T-fold compactification is to regard it as a formal tool for finding new massive supergravity theories. Compactification of massless higher dimensional type II supergravities on T-folds results in examples of lower dimensional massive supergravities, which in some cases were previously unknown [6, 8, 11]. The supergravities obtained in this manner can be thought of as the completion of the class of such theories obtained by compactification on geometric spaces (for examples based upon Scherk Schwarz reduction see [19 – 36]). In particular, these new sugras fit nicely into certain classifications of such theories, one of which is based on an $O(d,d)$ group of transformations [32, 11].

One of the most basic questions one would like to answer about these lower dimensional supergravity theories is how much supersymmetry they possess. This reduces to a question about the nature of the truncation of the higher dimensional fields which is used in the reduction. More precisely, the question is which of the higher dimensional supersymmetries are compatible with the different forms of truncation that one might use? This is the question we shall answer in part in this paper - how many supersymmetries are enjoyed by a theory obtained by T-fold compactification with a certain natural choice of truncation?

Our discussion will make it clear how to calculate the number of such supersymmetries associated with a large class of theories resulting from T-fold compactification. However, in the interests of clarity we will illuminate our considerations with the example of compactifications of type II string theory on T-folds which take the form of a T^d bundle over S^1, where the torus experiences a monodromy in $\mathrm{SO}(d,d,Z)$ around the base S^1. Similar compactifications have been considered in Refs. [6, 8, 11]. We will extend this work by incorporating extra terms originating from the NS B-field in the dimensional reduction and will then proceed to calculate the amount of supersymmetry that is manifest in the compactified theories obtained. To do this we will use and extend work in the literature [37 – 39] where the rules for the transformations of supersymmetry parameters under T-duality have been given. We will then use a modified version of the criterion for preservation of supersymmetry in dimensional reduction, phrased in terms of G-structures [40 – 73], to determine the number of supersymmetries manifest in the theory. The approach we will follow, based on the work of [37 – 39], will lead us to consider the supersymmetry of theories which result from compactification upon T-folds constructed using a certain minimal version of compactified supergravity. In particular, we will not consider cases arising from constructions where a coset reformulation of the theory obtained by reducing on the fibre of the T-fold is utilised in forming the overall compact space. See appendix A and later sections of the paper for more details.

The plan of this paper is as follows. In section 2 we shall introduce the T-fold backgrounds that we will be using as examples in the rest of the paper. We will then review dimensional reduction on such a background to illustrate the kind of massive supergravity that can be obtained in this manner. In section 3 we shall describe how to calculate the amount of supersymmetry associated with a class of theories obtained by T-fold reduction

using the examples of the previous section to illustrate our method. We will describe first how the usual arguments proceed in the case of geometric dimensional reduction before going on to describe how this analysis changes in the T-fold case. In section 4 we briefly conclude.

2. Examples of massive supergravities from T-fold reduction

2.1 Examples of T-fold vacua for dimensional reduction

The examples of duality-folds that we will consider are the class of T-folds which have been studied in [6, 8]. We start by considering a type II superstring theory on a T^d bundle over $M_{9-d} \times S^1$ (the discussion and notation of this subsection follows that of [25]). However, we are not going to take the trivial vacuum on this space. Instead we consider a situation where the various fields in the theory have a dependence on the S^1, which has coordinate $y \sim y + 2\pi$, of the following form,

$$\psi(y) = g(y)[\psi].\tag{2.1}$$

Here ψ is a general field which is taken to be independent of the T^d directions. We then make a Scherk-Schwarz ansatz [19, 20] for $g(y)$,

$$g(y) = \exp\left(\frac{yT}{2\pi}\right)\tag{2.2}$$

where T is in the Lie algebra of $\mathrm{SO}(d,d)$, which is a subgroup of the T-duality group associated with the T^d fibre. This ansatz ensures that a solution to the lower dimensional theory, obtained by compactification on this vacuum, is also a solution to the higher dimensional equations of motion up to the relevant approximations. As such it guarantees that we do not need to worry about such subtleties as compensators in the dimensional reduction. The resulting field configurations are not periodic on traversing the S^1. Instead there is a monodromy,

$$\mathcal{M}(g) = e^T.\tag{2.3}$$

This is simply an element of $\mathrm{SO}(d,d)$. In other words, when we traverse the circle once the fields are not identified in the usual way, but instead come back to themselves up to an $\mathrm{SO}(d,d)$ element of the T-duality group. In fact we should choose the monodromy to be within $\mathrm{SO}(d,d,Z)$ as this is the relevant symmetry subgroup of the full theory when massive states, which we have truncated, are included. Note that since we are making this Scherk-Schwarz ansatz we will only be able to consider elements of the full T-duality group, $O(d,d,Z)$, which are continuously connected to the identity in $O(d,d,R)$.

The resulting vacuum is an example of a T-fold [1]. The field configurations at $y = 0$ and $y = 2\pi$ are not simply related by diffeomorphisms and gauge transformations. A less trivial element of the T-duality group is required to transform them into one another. This vacuum is then non-geometric in the sense that there is not a globally well defined metric in ten dimensions. This is because the transition functions mix up the metric and NS-NS two form on the toroidal fibre when we go once around the circle.

2.2 Massive supergravities from dimensional reduction on T-folds

We will now describe how the low energy effective action associated with a dimensional reduction on one of the T-folds of the previous subsection is obtained. Many, but not all, of the terms in the reduced theory that we will present have been obtained elsewhere [6, 8]. However, it is useful to provide a discussion of the dimensional reduction here both to keep this paper reasonably self contained and also to complete the dimensional reduction of the ten dimensional NS-NS sector. To our knowledge some terms of this reduction, descending from the three form field strength, have yet to appear in the literature. Since this paper first appeared the relation between reductions of the type we are going to present here and so called 'twisted torus' reductions (see for example [11, 32]) has been demonstrated in [12].

For the sake of brevity we will consider a T^2 fibre, i.e. we will compactify on a T^2 bundle over S^1. The generalisation of this to the T^d case is straight-forward. The starting point for our dimensional reduction is the ten dimensional low energy effective action of type II superstring theory. Again, for the sake of brevity, and because it is common to both IIA and IIB, we only consider the NS-NS sector.

$$S_{10} = \int_{\mathcal{M}^{10}} dx \sqrt{-g} e^{-\phi} \left[R + (\partial \phi)^2 - \frac{1}{12} H^2 \right].$$ (2.4)

The simplest procedure to follow for the reduction is to first reduce on the T^2 fibre and then perform a further reduction on the base space incorporating our non-trivial twist. As such we require an effective eight dimensional action for the reduction of the above theory on the T^2 fibre in which the $O(2, 2)$ duality group is manifest. Such an eight dimensional action is given in the seminal paper by Maharana and Schwarz [74]. We will not repeat their calculation here but shall simply quote the results. They obtain,

$$S_{8D} = \int_{\mathcal{M}^8} d^8 x \sqrt{-g} e^{-\phi^{(8)}} \left[R^{(8)} + (\partial \phi^{(8)})^2 + \frac{1}{8} \text{tr}(\partial M^{-1} \partial M) - \frac{1}{4} \mathcal{F}^i_{\mu\nu} M^{-1}_{ij} \mathcal{F}^{\mu\nu j} - \frac{1}{12} H^2 \right],$$ (2.5)

where the three form field strength and eight dimensional dilaton are defined by,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} \mathcal{A}^i_\mu \eta_{ij} \mathcal{F}^i_{\nu\rho} + \text{cyclic permutations},$$ (2.6)

$$\phi^{(8)} = \phi - \frac{1}{2} \log \det G.$$ (2.7)

In these expression we have also defined,

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}$$ (2.8)

where B and G are the NS two form and metric on the fibre respectively. Finally, if we denote the two coordinate indices on the fibre as θ^i, the gauge potentials are given by

$$\mathcal{A}^i_\mu = G_{\mu\theta^i} \quad i = 1, 2,$$ (2.9)
$$\mathcal{A}^i_\mu = B_{\mu\theta^{i-2}} + B_{\theta^{i-2}\theta^j} \mathcal{A}^j_\mu \quad i = 3, 4,$$

and the field strengths are simply $\mathcal{F}^i = d\mathcal{A}^i$.

In this formulation, the various fields transform under an $O(2,2)$ transformation as follows,

$$M \to \Omega M \Omega^T \tag{2.10}$$

$$\mathcal{A}_\mu \to \Omega \mathcal{A}_\mu. \tag{2.11}$$

Here Ω is the $O(2,2)$ matrix, satisfying $\Omega^T \eta \Omega = \eta$, where in our conventions the invariant metric is given by,

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.12}$$

Given the $O(2,2)$ covariant form of the action in (2.5), we now make an ansatz for the dimensional reduction on the base of our T-fold. From the discussion in the previous subsection our ansatz takes the following form,

$$ds_8^2 = g_{\alpha\beta}^7(x^\gamma)dx^\alpha dx^\beta + e^{2\alpha}\left(dy + A_\alpha^B(x^\gamma)dx^\alpha\right)^2 \tag{2.13}$$

$$\phi^{(8)} = \phi^{(8)}(x^\alpha) \tag{2.14}$$

$$M = \Omega(y)M_0(x^\alpha)\Omega^T(y) \tag{2.15}$$

$$\mathcal{A}^i = \Omega_j^i(y)\mathcal{A}_\alpha^{0j}(x^\beta)dx^\alpha + \Omega_j^i(y)\mathcal{A}_y^{0j}(x^\beta)dy \tag{2.16}$$

$$B = \frac{1}{2}B_{\alpha\beta}(x^\gamma)dx^\alpha \wedge dx^\beta + B_\alpha^B(x^\gamma)dx^\alpha \wedge dy \tag{2.17}$$

where x^α are coordinates on the 7-dimensional non-compact space and

$$\Omega(y) = e^{\frac{y}{2\pi}T} \tag{2.18}$$

defines the monodromy. We can now perform the reduction on the base space.

The reduction of the Ricci scalar and dilaton terms to seven dimensions is unaffected by our duality twists. As such the expressions for these terms are the standard ones.

$$S_1 = \int_{\mathcal{M}^7} dx\sqrt{-g^7}e^{-\phi^{(7)}}\left[R^{(7)} - (\partial\alpha)^2 + (\partial\phi^{(7)})^2 - \frac{1}{4}e^{2\alpha}(F^B)^2\right]. \tag{2.19}$$

Here we have defined the usual seven dimensional dilaton $\phi^{(7)} = \phi^{(8)} - \alpha$.

The reduction of the remaining scalar kinetic terms does get modified by the duality twist. The resulting kinetic terms are the usual ones but the derivative involved is replaced by a covariant derivative with a non-trivial connection. In addition a potential for these fields is obtained. This potential and its properties have been discussed in some detail in [6]. The relevant terms are,

$$S_2 = \int_{\mathcal{M}^7} dx\sqrt{-g^7}e^{-\phi^{(7)}}\left[\frac{1}{8}\text{tr}\left(DM_0^{-1}DM_0\right) - \frac{1}{4(2\pi)^2}\text{tr}(T^T M_0^{-1}T M_0 + T^2)\right]. \tag{2.20}$$

Here we have defined a covariant derivative of M_0 as follows.

$$D_\alpha M_0 = \partial_\alpha M_0 - \frac{TM_0}{2\pi}A_\alpha^B - \frac{M_0 T^T}{2\pi}A_\alpha^B \tag{2.21}$$

where $M_0(x^\alpha)$ is defined by $M(x^\alpha, y) = \Omega(y)M_0(x^\alpha)\Omega(y)^T$.

The reduction of the vector field strength terms is a little more subtle. If we wish the result of the dimensional reduction to take an elegant form then we must ensure that we have made reasonable choices for our definitions of the seven dimensional fields [74, 32].

For example, consider the behaviour of the vector fields under an x^α dependent shift of the y coordinate, $dy \to dy' = dy + \partial_\alpha \omega^y dx^\alpha$.

$$\mathcal{A} \to \mathcal{A}' = \mathcal{A}'^{0i}_\alpha dx^\alpha + \mathcal{A}'^{0i}_y \left(dy + \partial_\alpha \omega^y dx^\alpha \right). \tag{2.22}$$

Thus we find that, while $\mathcal{A}^{0i}_y$ is unchanged by such a transformation, $\mathcal{A}^{0i}_\alpha$ is not a gauge invariant definition of a seven dimensional field.

$$\mathcal{A}^{0i}_\alpha \to \mathcal{A}'^{0i}_\alpha = \mathcal{A}^{0i}_\alpha - \partial_\alpha \omega^y \mathcal{A}^{0i}_y . \tag{2.23}$$

The only seven dimensional gauge field we wish to transform non-trivially under a shift of the base coordinate is the associated Kaluza Klein gauge field A^B. As such, instead of using $\mathcal{A}^{0i}_\alpha$ as a seven dimensional field we make a field redefinition to obtain a quantity that has the properties we desire.

$$\mathcal{A}^{7i}_\alpha = \mathcal{A}^{0i}_\alpha - A^B_\alpha \mathcal{A}^{0i}_y . \tag{2.24}$$

Having decided on a set of definitions for the seven dimensional fields we may proceed with the dimensional reduction of the vector field strength terms. These terms are affected by the duality twists and so we obtain modifications to the result obtained in an untwisted reduction which depend on the generator T. We find for the components of the field strength,

$$\mathcal{F}^j_{\alpha\beta} = \Omega^j_i (\mathcal{F}^{7i}_{\alpha\beta} + 2A^B_{[\beta} \partial_{\alpha]} \mathcal{A}^{0i}_y + \mathcal{A}^{0i}_y F^B_{\alpha\beta}) \tag{2.25}$$

$$\mathcal{F}^j_{\alpha y} = \Omega^j_i (\partial_\alpha \mathcal{A}^{0i}_y - \frac{T}{2\pi} A^B_\alpha \mathcal{A}^{0i}_y - \frac{T}{2\pi} \mathcal{A}^{7i}_\alpha) . \tag{2.26}$$

Here we have defined a field strength for $\mathcal{A}^{7i}_\alpha$ in the usual way. As always in such reductions the easiest way to proceed is to switch to using an orthonormal basis during the reduction and then return to a coordinate basis once the result has been obtained. We define the following vielbeins (barred indices denote those in an orthonormal basis).

$$e^\mu_\nu = \begin{pmatrix} e^{7\alpha}_{\bar\alpha} & -e^{7\beta}_{\bar\alpha} A^B_\beta \\ 0 & e^{-\alpha} \end{pmatrix} \quad e^{\bar\mu}_\nu = \begin{pmatrix} e^{7\bar\alpha}_\alpha & e^\alpha A^B_\alpha \\ 0 & e^\alpha \end{pmatrix} . \tag{2.27}$$

In these expressions, $e^{7\bar\gamma}_\alpha e^{7\bar\gamma}_\beta = g^7_{\alpha\beta}$, etc. in the usual manner. Using these definitions we then find the following.

$$-\frac{1}{4} \mathcal{F}^i M^{-1}_{ij} \mathcal{F}^j = -\frac{1}{4} \mathcal{F}^i_{\bar\alpha\bar\beta} M^{-1}_{ij} \mathcal{F}^{j\bar\alpha\bar\beta} - \frac{1}{2} \mathcal{F}^i_{\bar\alpha\bar y} M^{-1}_{ij} \mathcal{F}^{j\bar\alpha\bar y} \tag{2.28}$$

where,

$$\mathcal{F}^i_{\bar\alpha\bar\beta} = e^{7\alpha}_{\bar\alpha} e^{7\beta}_{\bar\beta} \mathcal{F}^i_{\alpha\beta} - 2e^{7\alpha}_{[\bar\alpha} e^{7\gamma}_{\bar\beta]} A^B_\gamma \mathcal{F}^i_{\alpha y} \tag{2.29}$$

$$\mathcal{F}^i_{\bar{\alpha}\bar{y}} = e^{7\alpha}_{\bar{\alpha}} e^{-\alpha} \mathcal{F}^i_{\alpha y}. \tag{2.30}$$

After a little algebra the following result is obtained for the reduction of the vector field strength terms.

$$S_3 = \int_{\mathcal{M}^7} dx \sqrt{-g^7} e^{-\phi^{(7)}} \left[-\frac{1}{4} \tilde{\mathcal{F}}^i_{\alpha\beta} M^{-1}_{0ij} \tilde{\mathcal{F}}^{j\alpha\beta} - \frac{1}{2} e^{-2\alpha} D_\alpha \mathcal{A}^{0i}_y M^{-1ij}_0 D^\alpha \mathcal{A}^{0j}_y \right]. \tag{2.31}$$

Here we have used the following definitions of field strengths and covariant derivatives.

$$\tilde{\mathcal{F}}^i_{\alpha\beta} = \tilde{\mathcal{F}}^{7i}_{\alpha\beta} + \mathcal{A}^{0i}_y F^B_{\alpha\beta} \tag{2.32}$$

$$\tilde{\mathcal{F}}^{7i}_{\alpha\beta} = \mathcal{F}^{7i}_{\alpha\beta} + 2A^B_{[\beta} \frac{T^i_j}{2\pi} \mathcal{A}^{7j}_{\alpha]} \tag{2.33}$$

$$D_\alpha \mathcal{A}^{0i}_y = \partial_\alpha \mathcal{A}^{0i}_y - \frac{T^i_j}{2\pi} A^B_\alpha \mathcal{A}^{0j}_y - \frac{T^i_j}{2\pi} \mathcal{A}^{7j}_\alpha. \tag{2.34}$$

This just leaves us with the dimensional reduction of the anti-symmetric tensor terms to perform. Despite the fact that the field strength H is itself invariant under the duality transformations we are using, the matrix T still enters this part of the reduction. The reason for this is that the field strength, while invariant overall, is made up of components which transform non trivially as can be seen from its definition (2.6).

As was the case for the reduction of the vector field part of the action we need to be careful to choose 'gauge invariant' definitions of our seven dimensional degrees of freedom. We find that B^B_α has the properties we desire but that $B_{\alpha\beta}$ does not. Therefore we define a new two form potential as follows.

$$B^7_{\alpha\beta} = B_{\alpha\beta} + A^B_{[\alpha} B^B_{\beta]}. \tag{2.35}$$

Then we can proceed with the dimensional reduction of these terms making use of our orthonormal basis as we did for the vector field strength piece. We find,

$$-\frac{1}{12} H^2 = -\frac{1}{12} H_{\bar{\alpha}\bar{\beta}\bar{\gamma}} H_{\bar{\alpha}\bar{\beta}\bar{\gamma}} - \frac{1}{4} H_{\bar{\alpha}\bar{\beta}\bar{y}} H_{\bar{\alpha}\bar{\beta}\bar{y}} \tag{2.36}$$

where,

$$H_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = e^{7\alpha}_{\bar{\alpha}} e^{7\beta}_{\bar{\beta}} e^{7\gamma}_{\bar{\gamma}} H_{\alpha\beta\gamma} - (e^{7\delta}_{\bar{\alpha}} e^{7\beta}_{\bar{\beta}} e^{7\gamma}_{\bar{\gamma}} A^B_\delta H_{y\beta\gamma} + 2 \text{ perms.}) \tag{2.37}$$

$$H_{\bar{\alpha}\bar{\beta}\bar{y}} = e^{7\alpha}_{\bar{\alpha}} e^{7\beta}_{\bar{\beta}} e^{-\alpha} H_{\alpha\beta\gamma} \tag{2.38}$$

and

$$H_{\alpha\beta\gamma} = \partial_\alpha B^7_{\beta\gamma} - \frac{1}{2} F^B_{\alpha\beta} B^B_\gamma + \frac{1}{2} H^B_{\alpha\beta} A^B_\gamma - \frac{1}{2} (\mathcal{A}^{7i}_\alpha + A^B_\alpha \mathcal{A}^{0i}_y) \eta_{ij} (\mathcal{F}^{7i}_{\beta\gamma} + \mathcal{A}^{0i}_y F^B_{\beta\gamma})$$
$$- (\mathcal{A}^{7i}_\alpha + A^B_\alpha \mathcal{A}^{0i}_y) \eta_{ij} A^B_{[\gamma} \partial_{\beta]} \mathcal{A}^{0i}_y \tag{2.39}$$

$$H_{\alpha\beta y} = H^B_{\alpha\beta} - \frac{1}{2} \mathcal{A}^{0i}_y \eta_{ij} (\mathcal{F}^j_{\alpha\beta} + 2A^B_{[\beta} \partial_{\alpha]} \mathcal{A}^{0i}_y + \mathcal{A}^{0i}_y F^B_{\alpha\beta}) - (\mathcal{A}^{7i}_{[\alpha} + A^B_{[\alpha} \mathcal{A}^{0i}_{|y|}) \eta_{ij} D_{\beta]} \mathcal{A}^{0i}_y. \tag{2.40}$$

$$- 8 -$$

In the above H^B is the field strength associated with B^B defined in the usual manner. After a little algebra we then find the following for the final piece of our seven dimensional action.

$$S_4 = \int_{\mathcal{M}^7} dx \sqrt{g^7} e^{-\phi^{(7)}} \left[-\frac{1}{12}(H^7)^2 - \frac{1}{4}e^{-2\alpha}(H^B_{\alpha\beta} - \frac{1}{2}\mathcal{A}_y^{0i}\eta_{ij}\tilde{\mathcal{F}}^j_{\alpha\beta} - \mathcal{A}^{7i}_{[\alpha}\eta_{ij}D_{\beta]}\mathcal{A}_y^{0i})^2 \right].$$
(2.41)

In this expression we have used the following definition of the seven dimensional three-form field strength.

$$H^7_{\alpha\beta\gamma} = \partial_\alpha B^7_{\beta\gamma} - \frac{1}{2}F^B_{\alpha\beta}B^B_\gamma - \frac{1}{2}H^B_{\alpha\beta}A^B_\gamma - \frac{1}{2}\mathcal{A}^{7i}_\alpha\eta_{ij}\tilde{\mathcal{F}}^j_{\beta\gamma} + \text{ cyclic perms.} \qquad (2.42)$$

Combining all the pieces we have obtained in this subsection we then obtain the following for the reduction of the NS-NS sector of type II string theory on this class of T-folds.

$$\begin{aligned}
S_7 = \int_{\mathcal{M}^7} \sqrt{-g^7} e^{-\phi^{(7)}} \bigg[& R^{(7)} - (\partial\alpha)^2 + (\partial\phi^{(7)})^2 - \frac{1}{4}e^{2\alpha}(F^B)^2 \\
& + \frac{1}{8}\text{tr}\left(DM_0^{-1}DM_0\right) - \frac{1}{4(2\pi)^2}\text{tr}(T^TM_0^{-1}TM_0 + T^2) \\
& - \frac{1}{4}\tilde{\mathcal{F}}^i_{\alpha\beta}M^{-1}_{0ij}\tilde{\mathcal{F}}^{j\alpha\beta} - \frac{1}{2}e^{-2\alpha}D_\alpha\mathcal{A}_y^{0i}M_0^{-1ij}D^\alpha\mathcal{A}_y^{0j} \\
& - \frac{1}{12}(H^7)^2 - \frac{1}{4}e^{-2\alpha}(H^B_{\alpha\beta} - \frac{1}{2}\mathcal{A}_y^{0i}\eta_{ij}\tilde{\mathcal{F}}^j_{\alpha\beta} - \mathcal{A}^{7i}_{[\alpha}\eta_{ij}D_{\beta]}\mathcal{A}_y^{0i}\,.)^2 \bigg].
\end{aligned}$$
(2.43)

In this action we have the following field strength and covariant derivative definitions.

$$F^B_{\alpha\beta} = 2\partial_{[\alpha}A^B_{\beta]} \qquad (2.44)$$

$$H^B_{\alpha\beta} = 2\partial_{[\alpha}B^B_{\beta]} \qquad (2.45)$$

$$\tilde{\mathcal{F}}^i_{\alpha\beta} = 2\partial_{[\alpha}\mathcal{A}^{7i}_{\beta]} + \mathcal{A}_y^{0i}F^B_{\alpha\beta} + 2A^B_{[\beta}\frac{T^i_j}{2\pi}\mathcal{A}^{7j}_{\alpha]} \qquad (2.46)$$

$$H^7_{\alpha\beta\gamma} = \partial_\alpha B^7_{\beta\gamma} - \frac{1}{2}F^B_{\alpha\beta}B^B_\gamma - \frac{1}{2}H^B_{\alpha\beta}A^B_\gamma - \frac{1}{2}\mathcal{A}^{7i}_\alpha\eta_{ij}\tilde{\mathcal{F}}^j_{\beta\gamma} + \text{ cyclic perms.} \qquad (2.47)$$

$$D_\alpha M_0 = \partial_\alpha M_0 - \frac{TM_0}{2\pi}A^B_\alpha - \frac{M_0 T^T}{2\pi}A^B_\alpha \qquad (2.48)$$

$$D_\alpha \mathcal{A}_y^{0i} = \partial_\alpha\mathcal{A}_y^{0i} - \frac{T^i_j}{2\pi}A^B_\alpha\mathcal{A}_y^{0j} - \frac{T^i_j}{2\pi}\mathcal{A}^{7j}_\alpha\,. \qquad (2.49)$$

It is easy to show that this action is that obtained by reducing the NS-NS part of the ten dimensional action on a three torus, supplemented by various mass and charge terms which are determined by the duality twist generator T. This is an example of the kind of massive supergravity we will consider in the next section. In particular, we will calculate the amount of supersymmetry this type of theory possesses.

In the low energy supergravity limit we have discussed here, the compactifications which have monodromies within the same $O(d,d,R)$ conjugacy class are equivalent and so give rise to the same reduced theory [6]. This point will tie in nicely with the discussion of supersymmetry that will appear in the next section. It should also be pointed out that the

above reduction implicitly uses the fact that the usual rules for flux quantisation in string theory are modified when the compact cycles being considered are nongeometric. There are scalar fields in the lower dimensional theory which originate from the components of the higher dimensional NS two form with indices on the compact space. For some of these fields, the fact that these degrees of freedom appear as lower dimensional fields which are able to vary in value depends crucially on the fact that the integral of the three-form field strength over certain cycles is not quantised in the usual manner.

3. Supersymmetry and T-fold reduction

This section is split into three main parts. Firstly we shall briefly review the standard discussion of how one determines the amount of supersymmetry associated with a geometric dimensional reduction. In the second subsection we will describe the transformation of various quantities under T-duality. Finally, we will use this information to describe how the standard discussion for the degree of supersymmetry of a dimensionally reduced theory is modified in the T-fold case.

A geometrical dimensional reduction proceeds in a series of steps. First one chooses a Riemannian manifold X, with metric and ansatze for the other fields, on which to compactify the extra dimensions. One then proceeds to rewrite the fields in the higher dimensional theory in a manner that respects the symmetries of this background. The fields are split up into representations of the subgroup of the full symmetry group of the theory that is preserved by the background. Up until this point nothing in the theory has actually changed. One has simply relabeled various fields in such a way as to make the rest of the dimensional reduction easier. The change to the content of the theory comes from the rest of the process. The extra dimensional dependence of the fields is expanded in a series of modes about the background and various parts of the expanded and rewritten higher dimensional fields, such as some of these modes, are discarded. The fields are then substituted into the higher dimensional action, compensators if needed are calculated, and the extra dimensions are integrated out to leave the lower dimensional theory. All of this has to be performed in a consistent manner so that every solution to the resulting low energy theory can be associated to a solution of the higher dimensional one, up to any approximations that are made in the truncation - which *defines* the lower dimensional theory. Furthermore, one might wish the truncation to be 'physical' in that the modes one discards in this process should describe fluctuations which are more massive than those which are kept. The classic example of this is Kaluza Klein reduction on a torus. In general, however, figuring out how physically to define the truncation is a highly non-trivial task.

Fortunately, for the Scherk-Schwarz reductions considered in the previous section a consistent truncation is easily defined. In fact, the ansatze for the fields already includes this truncation of the full possible field content of the theory.

The question we are interested in is how much supersymmetry is preserved by this sort of process (note we are interested in the amount of supersymmetry preserved by the lower dimensional *theory* as opposed to any specific vacuum of it). Supersymmetry, like any symmetry, describes the fact that if one mixes up the various fields in the theory

in a certain manner then one gets back to the same action as one began with. Given the above discussion, the crucial question becomes whether this mixing up of the fields is consistent with the expansion and truncation that has been made of the fields - no other procedure in dimensional reduction changes the theory and so no other procedure can break supersymmetry. Therefore, a supersymmetry will be preserved in a dimensional reduction if this process is such that the supersymmetry transformations only mix up parts of the higher dimensional fields which were kept in the truncated action with other parts that were kept. If this is not the case then the supersymmetry can not be preserved in the lower dimensional theory.

For commonly used geometrical compactifications and truncations there is a single mathematical rule which tells us which supersymmetries are preserved by dimensional reduction on direct product spacetimes. A supersymmetry is preserved if its parameter, ϵ, can be written as a product of an external and an internal spinor, the latter being a singlet under the structure group of the spin bundle associated to the internal manifold.[1] In other words, the internal part of the supersymmetry parameter should be globally defined and nowhere vanishing on the internal space in order for the associated supersymmetry to be preserved.

This rule is certainly true for the standard Kaluza Klein toroidal reduction where one truncates to zero modes - in this case all supersymmetry is then preserved. We shall show shortly that this rule is also true for the kind of Scherk Schwarz reductions we considered in the previous section when the monodromy is taken to lie in a geometric subgroup of $O(d,d)$. It is true for truncation to the massless modes on a Calabi-Yau reduction. In more complicated cases the same rule also holds; indeed in [65] a truncation was proposed for reductions on manifolds of SU(3) structure with non-vanishing intrinsic torsion which was constructed to obey this rule.

In the general case where the structure group of the frame bundle fills out the whole of $O(d)$, there are no singlets in the decomposition of the internal spinors under the structure group of the spin bundle. However, if the structure group only fills out part of $O(d)$ then singlets may exist.

To illustrate this consider dimensional reduction on a Calabi-Yau threefold, which is a six dimensional manifold with SU(3) structure. In fact Calabi-Yau manifolds also have SU(3) holonomy but this additional constraint is not important here as it relates to the amount of supersymmetry preserved by the four dimensional Minkowski space vacuum rather than the amount associated with the theory. Since the structure group of the frame bundle is SU(3) the structure group of the associated spin bundle is also SU(3) $\subset$ Spin(6) where Spin(6) is the double cover of SO(6). A spinor of Spin(6) is in the fundamental of SU(4) and so it is easy to see, if one imagines the internal spinor as a four component column vector, that the SU(3) structure will only leave one of the four independent components

---

[1]Since in a geometric compactification the internal manifold is Riemannian it has a metric tensor which constitutes an $O(d)$ structure for a d dimensional compact geometric space. The structure group of the frame bundle of the manifold is reduced from $Gl(d)$ to (a subgroup of) $O(d)$ in this case. Thus the frame bundle admits a principle sub-bundle which is the orthonormal frame bundle. If the manifold is spin this principle bundle has an associated spin bundle — sections of which form the spinor fields on the manifold.

of this spinor representation invariant. More formally the $\mathbf{4}$ of SU(4) decomposes under SU(4) $\supset$ SU(3) $\times$ U(1) as $\mathbf{4} = (\mathbf{1}, \mathbf{3}) + (\mathbf{3}, -\mathbf{1})$ where the first numbers in the brackets are the dimension of the relevant SU(3) representations and the second numbers are the U(1) charges. Thus we obtain one SU(3) singlet under this decomposition. Therefore, standard compactification on a Calabi-Yau threefold (or indeed any six dimensional manifold of SU(3) structure where the truncation used has been appropriately defined) preserves one quarter of the supersymmetry of the higher dimensional theory.

More generally, this condition for preservation of a certain number of supersymmetries in type II reductions would involve products of G-structures (see for example [65]). This is because the two higher dimensional supersymmetry parameters, $\epsilon_\pm$, can be decomposed using two different internal spinors $\chi_\pm$ which are associated to different G-structures. For example, if one follows this rule, continuing to examine the conditions for $N = 2$ supersymmetry in a reduction to four dimensions, the relevant condition on the internal manifold is that it should have SU(3) $\times$ SU(3) structure. The two internal spinors, $\chi_\pm$, which we require to be present by demanding $N = 2$ supersymmetry, each define an SU(3) structure. Locally, these two spinors need not be parallel and so the two SU(3) structures are different and locally define an SU(2) structure. However, if the SU(2) structure is globally defined then the dimensionally reduced action could instead be written as an $N = 4$ theory in four dimensions. Therefore, to avoid obtaining a lower dimensional theory that can be written in an $N = 4$ manner, the two spinors must be somewhere parallel. The SU(3) structure example mentioned above is a special case of SU(3)$\times$SU(3) structure where the two spinors are everywhere parallel (see for example [65]).

An important point to note here is that although in geometric reductions the higher dimensional supersymmetry parameters can be associated to a product of G-structures, this product is defined on a single spin bundle (and hence we can consider the intersection of the two structures). So even though the spinors $\epsilon_\pm$ are associated to different G-structures, they are sections of the same spin bundle. This is because there is only one orthonormal frame bundle with one associated spin bundle. This is one of the features of the discussion which will be modified in the forthcoming sections when we consider T-folds.

3.1 The behaviour of various quantities under T-duality

We wish to propose a modification to the rule for the amount of supersymmetry preserved by geometrical reductions, given above, to the case of non-geometric T-fold compactifications. However, before we can describe these modifications we need to know how various quantities, in particular the supersymmetry parameters, behave under T-duality transformations.[2]

For simplicity, in the rest of this paper we will consider the case where, in the lower dimensional theories we are interested in, supermultiplets containing fields coming from metric and B field components with a single fibre index are truncated. In terms of the massive supergravity presented in the last section for example this would correspond to the

---

[2]See appendix A for some comments about local reformulations of the theories under consideration which are relevant to this and following sections.

(consistent) truncation where the supermultiplets containing the calligraphic gauge fields (i.e. $\mathcal{A}_y^{0i}$ and $\mathcal{A}_\alpha^{7i}$) are set to zero. This truncation of the general case provides a vast simplification in the discussion that will be pursued in this section while still including many of the novel mathematical (non-) geometric structures that arise in these contexts. It would be of considerable interest to consider the more general case where these terms are not zero, however, and this will be pursued by the authors in future work.

Much of what we need to know about the behaviour of the supersymmetry parameters of type II theories under T-duality has been given in the literature [38, 37, 39]. In this work the relevant transformation rules were obtained by examining how the supersymmetry variations change under T-duality. Using the well known results for how the NS-NS sector transforms, such considerations are enough to fix the transformation properties of the spinor parameters.

In this paper we are interested in transformations whose generator, T, is not simply that of some geometric transformation such as a rotation. Of the $d(2d-1)$ generators of the group $\text{SO}(d,d)$, d^2 correspond to general linear coordinate transformations, and $d(d-1)/2$ correspond to B-field shifts. As such we shall follow [38] in only considering a subgroup of the full $O(d,d)$ T-duality group which contains the interesting new cases and which corresponds to symmetries of a given theory, rather than dualities which swap IIA and IIB. In fact we shall consider an $\text{SO}(d) \times \text{SO}(d)$ subgroup which includes an $\text{SO}(d)$ group which is simply the ordinary rotations. We consider this slightly larger than necessary subgroup as this choice makes the following discussion somewhat clearer. The generalisation to the $\text{SO}(d,d)$ case is straightforward, and will be discussed in appendix B. Our T-duality group elements will take the following form,

$$O = \frac{1}{2} \begin{pmatrix} S+R & S-R \\ S-R & S+R \end{pmatrix} \tag{3.1}$$

where S and R are $(d+b) \times (d+b)$ dimensional matrices which are related to $\text{SO}(d)$ matrices $\mathcal{S}$ and $\mathcal{R}$ as follows,

$$S = \begin{pmatrix} \mathcal{S} & \\ & 1_b \end{pmatrix} \qquad R = \begin{pmatrix} \mathcal{R} & \\ & 1_b \end{pmatrix}. \tag{3.2}$$

Here d is the dimension of the fibre and b is the dimension of the base. In this basis our $O(d,d)$ invariant metric is given by the following $2(d+b) \times 2(d+b)$ matrix,

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.3}$$

and so $O^T \eta O = \eta$ as required. The elements corresponding to ordinary rotations of the fibre are those where $S = R$. Note that O is analogous to Ω as introduced in section 2.2. However we have encoded the same information in higher dimensional matrices so that we can describe the relevant transformations as actions on points living in the entire compact space and not just the fibre. As such the dimension of O corresponds to the whole fibre bundle dimension, while Ω had dimension $2d$, and thus was associated only to the fibre.

– 13 –

We now consider the transformation rule for the metric and NS two form under O. As in the 2-dimensional case, the metric and two form on the fibre bundle (which has overall dimension $d + b$) can be combined into a matrix M which takes the same form as before, i.e.

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{3.4}$$

where now the blocks are $(d + b) \times (d + b)$ dimensional. Under T-duality $M \to OMO^T$. Using the explicit form for O we find the following transformation property for the inverse metric [38, 75]:

$$G^{-1} \to Q_- G^{-1} Q_-^T = Q_+ G^{-1} Q_+^T \tag{3.5}$$

In the above we have used the following definitions of the $(d + b)$ dimensional matrices Q:

$$Q_- = \frac{1}{2} \left[(S + R) + (S - R)(G + B) \right] \tag{3.6}$$
$$Q_+ = \frac{1}{2} \left[(S + R) - (S - R)(G - B) \right].$$

These equations imply two possible different transformations for the vielbein of the compact space under the transformation (3.1), either

$$e_{\bar{a}}^M \to \hat{e}_{(-)\bar{a}}^M = Q_{-N}^M e_{\bar{a}}^N \tag{3.7}$$
$$\text{or} \quad e_{\bar{a}}^M \to \hat{e}_{(+)\bar{a}}^M = Q_{+N}^M e_{\bar{a}}^N \ .$$

Here M is a $(d + b)$-dimensional spacetime index and $\bar{a}$ an orthonormal basis index. These two vielbeins are then related by a local Lorentz transformation as follows,

$$\hat{e}_{(+)\bar{b}}^{\ M} = \hat{e}_{(-)\bar{a}}^M \Lambda_{\bar{b}}^{\bar{a}} \quad \text{where} \tag{3.8}$$
$$\Lambda = e^{-1} Q_-^{-1} Q_+ e \tag{3.9}$$

Physically the situation is as follows. The vielbein $e_{\bar{a}}^M$ can be considered as arising from either the left $(+)$ or right $(-)$ moving sector of the string worldsheet. Under a T-duality transformation it transforms to either $\hat{e}_{(+)\bar{a}}^M$ or $\hat{e}_{(-)\bar{a}}^M$ depending on its worldsheet origin. These two local Lorentz frames are twisted relative to each other by an amount Λ. This generalises the statement that a single T-duality on a flat background is just parity on one of the world sheet sectors to the case of non-trivial backgrounds and more complicated transformations.

We see then from equation (3.7) that the two vielbeins transform differently, and both transformations are nonlinear. Thus from the point of view of discussing vielbeins on a T-fold we are left with the following picture. We define the left and right moving vielbeins separately. The transition functions seen by these vielbeins are a nonlinear realisation of $SO(d)$ transformations. Thus there are objects which resemble an orthonormal frame bundle. The differences to the geometric case lie in the fact that there are two 'orthonormal frame bundles' with (in general) different transition functions - one for the left and one

for the right moving sectors. In addition the transition functions on these objects act non-linearly. In the limit where we take $S = R$ (which corresponds to ordinary rotations) we find that $Q_- = Q_+ = S$ and the frame bundles reduce to the usual single orthonormal frame bundle with linearly acting transition functions.

Given these transformations for the orthonormal frames, how do the supersymmetry parameters transform under the T-duality element (3.1)? We denote the ten dimensional supersymmetry transformation parameters of our type II theories by $\epsilon_\pm$. We have followed [38] here in that the subscripts '$\pm$' denote the worldsheet sector to which the spinor is associated. We choose ϵ_- to have positive chirality in both type II theories which then fixes the chirality of ϵ_+ in both cases.

We define the spinor representation associated with Λ in the usual manner:

$$\Sigma_{\text{relative}}^{-1} \Gamma^a \Sigma_{\text{relative}} = \Lambda^a_{\ b} \Gamma^b. \tag{3.10}$$

The transformation rules for the supersymmetry parameters have then been found in the paper by Hassan, [38], to be,

$$\epsilon_- \rightarrow \Sigma_{\text{total}} \epsilon_- \tag{3.11}$$

$$\epsilon_+ \rightarrow \Sigma_{\text{total}} \Sigma_{\text{relative}} \epsilon_+ \tag{3.12}$$

where the explicit expression for Σ_{relative} is given by,

$$\Sigma_{\text{relative}} = 2^{-\frac{d}{2}} \sqrt{\frac{\det(\mathcal{Q}_- + \mathcal{Q}_+)}{\det \mathcal{Q}_-}} \left\{ 1 + \sum_{p=1}^{[d/2]} \frac{(-1)^p}{p!\, 2^p} \mathcal{A}^{i_1 i_2} \dots \mathcal{A}^{i_{2p-1} i_{2p}} \Gamma_{i_1 i_2 \dots i_{2p-1} i_{2p}} \right\}, \tag{3.13}$$

where $\mathcal{Q}_\pm$ are the $d \times d$ blocks of the matrices $Q_\pm$, which are associated to the fibre, and the quantities $\mathcal{A}^{ij}$ are components of the following matrix:

$$\mathcal{A}^{ij} = \left[(1_d - \mathcal{S}^{-1}\mathcal{R})^{-1} (1_d + \mathcal{S}^{-1}\mathcal{R}) + \mathcal{B} \right]^{-1}_{ij} \tag{3.14}$$

where the $\{ij\}$ indices are raised by matrix inversion, and $\mathcal{B}$ is the B-field on the d dimensional fibre.

Now the work by Hassan is concerned solely with local considerations, and therefore the overall rotation on the spinors, Σ_{total}, can be set to the identity using the local Lorentz symmetry. This makes the spinor ϵ_- invariant under the T-duality transformations. However, in our case we wish to use the spinor transformations as transition functions. Therefore, we are no longer free to make the choice $\Sigma_{\text{total}} = 1$. We must specify the total action of the transformations on the spinors up to a conjugacy class, as this is the degree to which the transition functions are defined given the presence of local Lorentz symmetry. This should be compared with the similar situation which exists in the geometric case. There $\Sigma_{\text{relative}} = 1$, and Σ_{total} implements the rotation on the spinors associated to the rotation of the underlying frame.

There are a number of ways in which we can isolate the overall rotation on the spinors. The structure of the 'orthonormal frame bundles' discussed above makes the correct choice relatively clear - one should pick the overall rotation so that there is a correspondence

between the action on the orthonormal frames and the action on the spinors which preserves group structure. In other words the left and right moving spin bundles should be 'associated to' the left and right moving frame bundles. One consequence of this link is that, due to the nonlinear and B-dependent nature of the transition functions on the frame bundles, the transition functions on the spin bundles will in general depend on G and B as well as S and R. There are a number of symmetries we expect the spinor transition functions to obey, which follow from the association with the orthonormal frame bundles. Firstly, in the limit of flat space with no B field (i.e. $G + B = \mathbb{1}$), Q_+ and Q_- are simply the linearly acting rotations R and S respectively. Therefore, we expect the transformations on the spinors to be simply given by the spinor representations of R and S in that case. Secondly, we notice that for a general background there is a symmetry $Q_+ \leftrightarrow Q_-$ if we take $B \to -B$ and exchange $S \leftrightarrow R$. This symmetry exchanges the left and right moving world sheet sectors and so the two frame bundles. Therefore, we expect this same symmetry to be present in the spinor transition functions; in particular it should exchange the the transition functions for ϵ_+ and ϵ_-.

In practical terms, how do we actually construct the overall rotation? The simplest method is the following. Consider a background that is a T^d fibration over an S^1 base with some monodromy associated with the non-contractible loop of the base. We take the monodromy to lie in the non-trivial subgroup $\mathrm{SO}(d) \times \mathrm{SO}(d)$ that we have been discussing. As we traverse the S^1, starting at $y = 0$ and ending at $y = 2\pi$, the orthonormal frames $e_\pm$ undergo field-dependent transformations implemented by $Q_\pm$. Due to the monodromy in the background these frames do not in general come back to themselves at $y = 2\pi$, but are related to the original frames by some T-duality transformation. Now consider parallel transporting the spinors $\epsilon_\pm$ around paths where all of the coordinates are constant except for y. The left moving spinor is transported using the spin connection constructed from the left moving vielbein, e_+, and similarly for the right moving sector. We find that when we compare the two spinors with themselves at $y = 0 = 2\pi$ they have undergone a rotation. This rotation is defined up to a conjugacy class due to the local Lorentz invariance on the coordinate patch. We get different rotations on the left and right moving spinors and these have precisely the properties that we require for those associated to the monodromy element we have picked - they obey all of the symmetries mentioned above (this is guaranteed because our parallel transport operators involve spin connections derived from the two vielbeins $e_\pm$, and the vielbeins satisfy these properties) and, in particular, give the correct relative rotation, $\Sigma_{\mathrm{relative}}$, given in (3.13), which was derived by Hassan using another method.

By carrying out this construction for an arbitrary monodromy within our $\mathrm{SO}(d) \times \mathrm{SO}(d)$ subgroup we can find all the previously unknown transition functions on the supersymmetry parameters. These can then be used to construct the spin bundles associated to a given T-fold. We emphasise that the construction we have outlined here need not be based upon the T-fold we are finally interested in. It constitutes instead merely a trick for finding transition functions which, once we have them, can be used in any appropriate T-fold context we desire.

Example. To give a concrete illustration of the above discussion we will now present the details for a simple example. We shall examine the case where we have a T^2 fibre on an S^1 base with a $U(1) \times U(1)$ monodromy in a 'nontrivial' subgroup of $SO(2, 2, Z)$.

First, we label the coordinates on the T^2 fibre by z^i, $i = 1, 2$, and the coordinate on the base S^1 by $y \in [0, 2\pi]$. We will also use the combined coordinate z^A, $A = 1, 2, 3$, where $z^3 \equiv y$. In these coordinates, the metric on the fibre at $y = 0$ is given by the usual torus metric:

$$\mathcal{G}_0 = \frac{\rho_2}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \tag{3.15}$$

where $\tau = \tau_1 + i\tau_2$ is the complex structure modulus, and ρ_2 is the volume modulus. It is convenient to combine the volume modulus into a complex field $\rho = \rho_1 + i\rho_2$, where ρ_1 is related to the B-field at $y = 0$ as follows,

$$\mathcal{B}_0 = \begin{pmatrix} 0 & \rho_1 \\ -\rho_1 & 0 \end{pmatrix} \tag{3.16}$$

We recall that we are considering the truncation where there are no off-diagonal terms in the metric or B-field between the fibre and the base, i.e. the components which have one base and one fibre index are set to zero. Therefore, the metric on the fibre bundle at $y = 0$ is given by

$$ds^2 = \frac{\rho_2}{\tau_2} \left| dz^1 + \tau dz^2 \right|^2 + e^{2\alpha} dy^2 \tag{3.17}$$

where α is a function of the lower-dimensional space. From the form of the metric above, a natural vielbein and inverse vielbein to take at $y = 0$ are

$$e_A^{\bar{a}} = \tau_2^{-1/2} \begin{pmatrix} \rho_2^{1/2} & 0 & 0 \\ \rho_2^{1/2}\tau_1 & \rho_2^{1/2}\tau_2 & 0 \\ 0 & 0 & e^{\alpha}\tau_2^{1/2} \end{pmatrix} \qquad e_{\bar{a}}^A = \tau_2^{1/2} \begin{pmatrix} \rho_2^{-1/2}\tau_2 & 0 & 0 \\ -\rho_2^{-1/2}\tau_1 & \rho_2^{-1/2} & 0 \\ 0 & 0 & e^{-\alpha}\tau_2^{1/2} \end{pmatrix} \tag{3.18}$$

Now the torus fibre experiences a monodromy around the base S^1. To illustrate our method we choose this monodromy (and so the transition function action we are calculating) to take the form

$$O = \begin{pmatrix} \cos(a) & 0 & 0 & 0 & -\sin(a) & 0 \\ 0 & \cos(a) & 0 & \sin(a) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\sin(a) & 0 & \cos(a) & 0 & 0 \\ \sin(a) & 0 & 0 & 0 & \cos(a) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.19}$$

which corresponds to the following rotation matrices $\mathcal{S}$ and $\mathcal{R}$ (see (3.1)),

$$\mathcal{S} = \mathcal{R}^T = \begin{pmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{pmatrix} \tag{3.20}$$

– 17 –

If we compare with Ref. [6] this monodromy lies in the elliptic conjugacy class of $SL(2)_\rho$. In a stringy application we should take $a = \pi/2$ in order to obtain a non-trivial monodromy within $SL(2, Z)_\rho$. We now require a y-dependent O which implements this monodromy as we go from $y = 0$ to $y = 2\pi$. The obvious choice is to replace $a \rightarrow ay/2\pi$ in the above expressions. However, it is more convenient to choose the following simpler expressions for the base space dependent matrices,

$$\mathcal{S}(x(y)) \equiv \mathcal{S}(x) = \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} 1 & -x \\ x & 1 \end{pmatrix}, \qquad \mathcal{R}(x) = \mathcal{S}(x)^T \tag{3.21}$$

where $x(y) = \tan(ay/2\pi)$ and so $x = 0$ and $x = \tan(a)$ are the endpoints of the loop in the base space (we assume $a \leq \pi/2$, so that the parameterisation makes sense). Note that using x is simply a coordinate choice for the base, and the final answers will not depend on this choice (we will see this explicitly from our expressions later on). These x-dependent matrices allow us to calculate the fibre metric and B-field at arbitrary points on the base space using (2.10), i.e.

$$M(x) = O(x) \; M_0 \; O(x)^T \tag{3.22}$$

where M_0 is the matrix given in (3.4) constructed from the fibre bundle metric (3.17) and B-field (3.16). Then using all of these x-dependent quantities we can calculate the matrices $Q_+(x)$ and $Q_-(x)$ which describe the transformations of the left and right moving frames as a function of the S^1 base's coordinate:

$$e_{(\pm)\bar{a}}^{A}(x) = Q_\pm{}^A{}_B(x)e_{\bar{a}}^B \tag{3.23}$$

where $e_{\bar{a}}^B$ is the inverse vielbein at $y = 0$ given in (3.18), and $Q_\pm(x)$ are constructed from $\mathcal{S}(x)$, $\mathcal{R}(x)$, and $M(x)$.

We now use the expressions for $e_{(\pm)}(x)$ to calculate the components of the two associated three dimensional spin connections. In fact, we do not need to determine all of the components of the connections but simply those which appear in the expressions governing parallel transport of the spinors around the relevant closed loops. Now the spinors $\epsilon_\pm$, which arise from the left and right moving sectors on the world-sheet, split up into external 7-dimensional spinors, $\eta_\pm$, and internal 3-dimensional spinors, $\chi_\pm$. Since we are interested in the internal space, and because we have made the simplification mentioned at the beginning of section 2, we will only need to deal with the spinors $\chi_\pm$ from now on. The equations for parallel transport around a path where the fibre coordinates are constant are

$$\nabla_x^{(\pm)}\chi_\pm = \partial_x\chi_\pm + \frac{1}{4}\omega_{(\pm)x}^{\bar{a}\bar{b}}\Gamma_{\bar{a}\bar{b}}\chi_\pm = 0 \tag{3.24}$$

where summation is assumed over the orthonormal indices $\bar{a}, \bar{b}$. It can easily be shown that the expression for the spin connection components is

$$\omega_{(\pm)x}^{\bar{a}\bar{b}} = \sum_A e_{(\pm)}{}^{A[\bar{a}} \; \partial_x e_{(\pm)A}^{\bar{b}]}. \tag{3.25}$$

Using the expressions (3.23) for the left and right moving vielbeins as functions of x this becomes

$$\omega_{(\pm)x}^{\bar{a}\bar{b}} = -e_A^{[\bar{a}} \left(\partial_x (Q_{(\pm)}^{-1}(x))^A{}_B (Q_{(\pm)}(x))^B{}_C \right) e^{\bar{b}]C} \tag{3.26}$$

with summation understood over A, B, C. Notice that the spin connection is now written in terms of the original vielbein at $x = 0$. Using this equation we find that the only non-vanishing spin connection components are

$$\omega_{(-)x}^{\bar{1}\bar{2}} = -\omega_{(+)x}^{\bar{1}\bar{2}} = -\frac{2x^2(x^2-1)|\rho|^2 + 8x^3\rho_1 - x^4 + 4x^2 + 1}{|1+x\rho|^2(4x^2|\rho|^2 - 4x(x^2-1)\rho_1 + x^4 - 2x^2 + 1)}. \tag{3.27}$$

The simple relation between ω_+ and ω_- is due to the relationship $\mathcal{S} = \mathcal{R}^T$ for this monodromy. Note also that this spin connection component does not depend on the complex structure modulus of the torus. This is because the monodromy is in the $SL(2)_\rho$ subgroup of $SO(2,2)$. We now have the required information to explicitly parallel transport the spinors around our paths with $z^1 = z^2 = $ constant. The resulting spinors at $x = \tan(a)$ are given by

$$\chi_{\pm} = \exp\left(-\frac{1}{2}\int_0^{\tan a} \omega_{(\pm)x}^{\bar{1}\bar{2}}\Gamma_{\bar{1}\bar{2}} \, dx\right)\chi_{\pm}^0 \tag{3.28}$$

where $\chi_{\pm}^0$ are the spinors at $x = 0$. Note that from the above expression it is clear, given that the spin connection contains one x derivative, that we could have chosen any parameterisation for the closed loop and obtained the same answer. From our explicit expression for $\omega_{(\pm)x}^{\bar{1}\bar{2}}$ we obtain

$$-\frac{1}{2}\int_0^{\tan(a)} \omega_{(\pm)x}^{\bar{1}\bar{2}}\Gamma_{\bar{1}\bar{2}}dx = \pm\frac{1}{2}\left\{-\arctan\left(\frac{\rho_1}{\rho_2}\right) + \arctan\left(\frac{|\rho|^2\tan(a) + \rho_1}{\rho_2}\right)\right.$$
$$\left. + \arctan\left(\frac{2\rho_2\tan(a)}{\tan^2(a) - 1 - 2\tan(a)\rho_1}\right)\right\}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.29}$$

where we have chosen the following realisation for the Γ matrices: $\Gamma_{\bar{1}} = \sigma_1$, $\Gamma_{\bar{2}} = \sigma_3$, where σ_i are the usual Pauli matrices. The exponential of this matrix then gives the explicit matrix which takes $\chi_{\pm}^0$ to $\chi_{\pm}^{2\pi}$ for this monodromy. Since we therefore now have the action of the monodromy on both kinds of spinor we have also determined the overall rotation we have been looking for. Clearly the matrix multiplying χ_+^0 is the inverse of that multiplying χ_-^0. This is simply a consequence of the choice of monodromy we have made in this example. Despite this simple relation, it should be noted that formally the two transition functions associated to $\chi_{\pm}$ live in the structure groups of two different spinor bundles.

Note that if we take the flat space limit at $y = 0$, i.e. take $\mathcal{G}_0 + \mathcal{B}_0 = \mathbb{1}$ (which corresponds to taking $\rho = \tau = i$, although note that the τ modulus does not actually appear, so its value doesn't make a difference here), then the factor multiplying the constant matrix in (3.29) becomes $\mp a/2$. This is $\mp 1/2$ times the angle which appears in the monodromy matrix. This makes sense because in the flat space limit $Q_- = S$, $Q_+ = R$ and so the spinors transform via the spin representation of these rotation matrices. From (3.29) we see that another expected feature of these transformations also holds, namely that taking $a \to -a$ and $\rho_1 \to -\rho_1$ exchanges the transformation matrices for χ_- and χ_+.

As a brief aside before we move on to discuss the preservation of supersymmetry, we note that in the 2-torus case, the matrix multiplying $\chi_\pm^0$ in (3.28) can be written in a nicer form as the spin representation of the following matrix

$$A_\pm = \begin{pmatrix} \frac{1}{\sqrt{\det(\mathcal{Q}_\pm)}} \tilde{e}^{-1} \mathcal{Q}_\pm \tilde{e} & 0 \\ 0 & 1 \end{pmatrix} \tag{3.30}$$

where $\mathcal{Q}_\pm$ is evaluated at $x = \tan(a)$ and $\tilde{e}$ is the 2×2 part of the vielbein e. It turns out that in the $O(2,2)$ case, these matrices are orthogonal. In other words,

$$\Sigma(A_\pm) = \exp\left(-\frac{1}{2} \int_0^{\tan a} \omega_{(\pm)_x}^{\bar{1}\bar{2}} \Gamma_{\bar{1}\bar{2}} \, dx \right) \tag{3.31}$$

where Σ denotes the spin representation, obtained in the usual way. Similar comments do not hold in the case of higher dimensional fibres however.

In any case we now set aside our example and return to the general discussion to show how to determine the amount of supersymmetry preserved by appropriately defined T-fold reductions.

3.2 'G-structures' and supersymmetry in the non-geometric case

We are now going to describe a proposal for a simple rule for determining the amount of supersymmetry preserved by T-fold reductions of the form of a T^d bundle over S^1 when a sensible truncation is defined in the dimensional reduction process. Our discussion generalises in an obvious way to more complicated T-fold reductions. As discussed earlier, we are interested in describing the supersymmetry preserved by the compactified *theory* rather than by any particular lower dimensional vacuum. Therefore, we will follow the analogue of the G-structures story that we outlined at the start of this section. In addition we repeat that we are considering the case where supermultiplets with fields originating from higher dimensional metric and B field components with a single fibre index have been truncated and that we are not considering the 'non-minimal' cases discussed in the appendix A.

The rule we propose for the amount of supersymmetry preserved by T-fold reduction in fact follows along fairly similar lines to the geometric case if the situation is phrased in terms of structure groups of spin bundles. As in the geometric case we decompose the spinor parameters into sums of internal and external pieces. We then consider how the internal pieces of these spinors transform under the structure groups of the relevant spin bundles (of which there are now two). We propose that the number of singlets in the decomposition of each internal spinor into representations of the relevant structure group then determines how many lower dimensional supersymmetries are associated to it in the case of an appropriately defined truncation.

However, there are crucial differences between the non-geometric and geometric cases. One obvious difference is that on T-folds the transition functions on the spinor super-symmetry parameters are generally B and G dependent, as we have seen explicitly in the previous section. This point deserves a little further discussion. At a first glance one might

naively assume that this fact, coupled with the above comments, means that the amount of supersymmetry of the lower dimensional *theory* is a moduli dependent quantity! This clearly can not be the case. In fact since we are talking about the theory rather than about any particular lower dimensional vacuum (and so moduli values) the correct amount of supersymmetry is the minimum that is obtained when arbitrary values of the moduli are allowed.

We could however create a different lower dimensional theory if a consistent truncation were available where some of the lower dimensional fields were set to certain constant values (indeed we have already done something very similar with the fields we have already truncated). This could conceivably result in a truncated theory with more supersymmetry than its parent theory. Such phenomena are well known within supersymmetric compactifications and it should not come as a surprise that examples could also appear in this context.

The fact that we have two different spin bundles with different transition functions on them is major difference between the case at hand and that of geometric compactification. For example, the existence of two globally well defined, nowhere parallel internal spinors in the geometric case would imply the existence of 4 supersymmetries in the dimensionally reduced theory if the usual rule where to hold (i.e. two from each of ϵ_+ and ϵ_-). However, in the T-fold case, the same need not necessarily be true. The two internal spinors could be sections of different spin bundles. This would lead to us obtaining one lower dimensional supersymmetry from each higher dimensional spinor parameter - giving us a total of two rather than four.

Let us examine now the examples of section 2 where we performed a Scherk-Schwarz reduction and truncation. Do these examples obey the rule we have proposed for the number of supersymmetries in the lower dimensional theory? In fact, this is a relatively easy question to answer.

We have, by the nature of our reduction ansatz (2.1), truncated the higher dimensional theory to consider only field configurations of the form

$$\psi(y) = g(y)[\psi_0] \,. \tag{3.32}$$

Which supersymmetries are compatible with this truncation? Clearly from our earlier discussion only those supersymmetry transformations which, when applied to field configurations of this form give back another field configuration also in this form are compatible. Applying a supersymmetry transformation to a field configuration of this form we find the following,

$$(1 + \delta_\epsilon)\psi(y) = (1 + \delta_\epsilon)g(y)[\psi_0] \tag{3.33}$$

$$= g(y)[(1 + \delta_{\Sigma^{-1}\epsilon})\psi_0] \,. \tag{3.34}$$

Thus only if $(1 + \delta_{\Sigma^{-1}\epsilon})\psi_0 = \psi_0'$, where ψ_0' is independent of y, do we get back to a field configuration of the same form. Given Σ's dependence on y in our examples this means that only supersymmetries of the form $\epsilon = \Sigma\epsilon_0$, where ϵ_0 is not dependent on the internal coordinates, are compatible with the truncation. However, such spinors are not

globally well defined unless $\Sigma\epsilon_0 = \epsilon_0$. In other cases the spinor ϵ will be double valued at $y = 0, 2\pi$. Thus as our rule states, the supersymmetries which are preserved in Scherk-Schwarz reduction arise from internal spinors which are singlets under the monodromy (and so the structure group).

Clearly there is a lot of interesting (non)-geometrical structure in the above. In addition we have now answered the question we set out to address: we know how to calculate the amount of supersymmetry preserved by certain T-fold reductions.

Some examples. Let us illustrate our discussion by working out the amount of supersymmetry associated with the appropriate truncations (dropping the supermultiplets containing the calligraphic fields of section 2) of the massive supergravity theories discussed earlier in the paper. We therefore consider the case where we have a T^d fibration over an S^1 base. The structure group of the spin bundles is simply given by the monodromy e^T which is introduced into the background. Therefore, in all of our cases the monodromy simply constitutes a single operator which then lies in a U(1) subgroup of $O(d, d)$. One can then use the discussion of this section to determine, for any given monodromy, how this U(1) acts on the left and right moving spinors. For the case of a T^2 fibre the **16** of SO(9, 1) breaks up into $(\mathbf{8}, \mathbf{2})$ representations of $SO(6, 1) \times SO(3)$. Now clearly because the spinor of SO(3) is in the fundamental of SU(2), a U(1) structure group will leave none of the degrees of freedom of the **2** invariant. For general values of B and G the monodromy will result in a non-trivial structure group for both spin bundles, as can be seen from our example of the previous subsection. Therefore in general we will be left with none of the higher dimensional supersymmetries being present in the lower dimensional theory. Thus in the general case the resulting 7-dimensional theory, as presented in section 2, has no supersymmetry.

However, as discussed earlier, if one takes a truncation of the lower dimensional theory then it is possible that this truncated theory could contain more supersymmetry than its parent. Let us consider the particular example presented in section 3.1 where we chose a certain monodromy for the T^2 fibre (3.19) and see if this occurs in this case. By considering the scalar potential (2.20) associated to this monodromy, one finds that a consistent truncation can be made by setting $\rho = i$, as this minimises the potential for the relevant moduli. However in this case, even after such a truncation, we see from our expressions at the end of section 3 that the monodromy still acts on both types of spinor in a non-trivial way. Thus for this particular choice of monodromy the truncated theory also has no supersymmetry and the phenomenon of enhancement of symmetry on truncation does not occur.

For the case of a T^3 fibre, the story for the general case is slightly different. The **16** of SO(9, 1) breaks up into $(\mathbf{4}, \mathbf{2}) + (\mathbf{4'}, \mathbf{2'})$ representations of $SO(5, 1) \times SO(4)$. Now the two Weyl spinors of SO(4) that arise here transform under different SU(2) subgroups and so one of these will remain invariant if we arrange matters so that the U(1) is a subgroup of the other SU(2). The analysis of the other SU(2) will follow as in the case of the T^2 fibre. Therefore, in general half the supersymmetries associated to ϵ_+ and ϵ_- are preserved, and so half of the possible supersymmetries in the six dimensional theory are preserved, so we obtain either (1, 1) or (2, 0) supersymmetry in 6 dimensions.

As before, a consistent truncation of the lower dimensional theory could result in more supersymmetry with different numbers of supersymmetries coming from the two higher dimensional supersymmetry parameters for certain choices of monodromy.

Similar analyses can be pursued for all higher dimensional fibres. Clearly the arguments described in this section can also easily be extended to deal with more general T-fold constructions, where the base is not simply S^1.

4. Conclusions

In this paper we have considered the dimensional reduction of type II supergravity theories on T-folds. We began by briefly discussing the dimensional reduction of higher dimensional actions to obtain lower dimensional massive supergravity theories. We then went on to discuss some underlying (non-) geometric structures associated with these compactifications in order to calculate how much supersymmetry the lower dimensional theories possess.

We showed that associated with these spaces there are two 'orthonormal frame bundles', one associated with each of the left and right moving sectors of the string worldsheet. These have non-linearly acting, field dependent transition functions on them. Associated to these orthonormal frame bundles, in some sense, are two spin bundles. Due to the nature of the action of T-duality on the relevant spinors these also have field dependent transition functions. By examining the subset of the full field dependent representation of $SO(d)$ that these transition functions take in any one case, we were able to identify any spinors that are invariant under the structure group of the bundle of which they are sections. As in the geometric case this information then told us how much supersymmetry to expect in an appropriately defined dimensionally reduced theory. All of the structure we described reduces to the usual structure of a single orthonormal frame and spin bundle etc. when the structure group is taken to lie within a geometric subgroup. We finally illustrated our discussion by calculating the amount of supersymmetry associated with truncations of various massive supergravity theories that have been presented in the literature and in the early part of this paper. It should be noted that we have been concerned in this paper with the amount of supersymmetry associated to the compactified *theory* rather than any one particular lower dimensional vacuum.

There is clearly much work still to be done on the investigation of various properties of T-folds. One of the more interesting directions for future work would be to look for connections between the structure presented here and mathematical frameworks such as generalised complex geometry.

Acknowledgments

A. Coset reformulations

In this paper when we have considered type II supergravity compactified on a torus we have used the formulation of this theory with the minimum number of degrees of freedom in it. There are various other formulations of these theories. Indeed two such reformulations are given in the paper of Maharana and Schwarz [74]. These descriptions of the theory essentially constitute introducing extra degrees of freedom into the theory as well as extra auxiliary gauge symmetries in order to remove them again. This kind of procedure might be carried out, for example, in order to linearise the action of the T-duality transformations acting on the fields. Locally these reformulations of the theory are all physically equivalent. One can recover the description of the theory used in this paper from one of these coset formulations merely by stipulating a specific gauge choice for the auxiliaries. Globally, however, the formulations can differ. The presence of the auxiliary gauge symmetries can lead to a richer global structure than is present in the minimal version of the theory. One could, for example, have Wilson lines in these gauge fields. It would then be impossible to choose a gauge to recover the minimal formulation on every coordinate patch at the same time.

In this paper we only consider the standard formulation and the supersymmetry of compactifications on T-folds which are constructed within this framework. It would clearly be of interest to study the coset reformulations of this theory in this context as well. However, this is beyond the scope of this paper. It should be pointed out that our results do apply to a subset of the configurations that can result from these other formulations. The relevant subset is that where the global structure of the auxiliary gauge fields is such that we can make the gauge choice to restore the minimal formulation simultaneously on every coordinate patch.[3]

B. Extension to more general $SO(d, d)$ monodromies

In the main body of this paper we have considered monodromies which lie in a $d(d-1)/2$-dimensional subgroup of $SO(d, d)$ and take the "non-trivial" form given in (3.1). However, we can consider more general elements of $SO(d, d)$ by combining these non-trivial twists with general coordinate transformations and B-shifts. In the case $d = 2$ we can generate a general element of $SO(2, 2)$ by the following product of group elements,

$$O_{\text{total}} = \Lambda B O \tag{B.1}$$

where Λ is a coordinate transformation matrix, B defines a B-shift and O is a non-trivial $SO(2) \times SO(2)$ matrix of the form (3.1). Explicitly, the B-shifts take the following form,

$$B_i = \begin{pmatrix} 1 & 0 \\ b_i & 1 \end{pmatrix}$$

---

[3]We would like to thank Chris Hull for pointing out the importance of these reformulations of the theory in this context.

and the matrix Λ is given by

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & (\lambda^T)^{-1} \end{pmatrix}$$

where λ is the 2×2 matrix associated to the coordinate transformation. We now consider the action of the element O_{total} on the spinors ϵ_+ and ϵ_-. Firstly, we note that B-shifts have no effect on either spinor parameters, but the value of B does enter the matrices $\Sigma^{(\pm)}$, which are the spinor transformation matrices associated to the non-trivial element O. So the effective action on the spinors $\epsilon_\pm$ is

$$\epsilon_+ \to \mathcal{L} \, \Sigma^{(+)} \epsilon_+ \tag{B.2}$$

$$\epsilon_- \to \mathcal{L} \, \Sigma^{(-)} \epsilon_- \tag{B.3}$$

where $\mathcal{L}$ is the spin representation of the coordinate transformation Λ (note that the spinors will only "see" the rotation part of the matrix Λ). For $d \geq 3$ the situation is slightly different as we can generate all elements of $\mathrm{SO}(d,d)$ from B-shifts, coordinate rotations (as opposed to general coordinate transformations) and non-trivial twists. This can be seen by considering the Lie algebra elements associated to each of the transformations, and by calculating their commutators. One finds that a general element of $\mathrm{SO}(d,d)$, $d \geq 3$, can be written as

$$O_{\text{total}} = L O_3 B_3 O_2 B_2 O_1 \tag{B.4}$$

where L is a coordinate rotation. The corresponding spinor transformations are then

$$\epsilon_+ \to \mathcal{L} \, \Sigma_3^{(+)} \Sigma_2^{(+)} \Sigma_1^{(+)} \epsilon_+ \tag{B.5}$$

$$\epsilon_- \to \mathcal{L} \, \Sigma_3^{(-)} \Sigma_2^{(-)} \Sigma_1^{(-)} \epsilon_- \tag{B.6}$$

Note that the fact that B changes is important in these transformations, as a different value of B will enter each Σ_i. With this information we could now tackle a T-fold with monodromy in the full connected subgroup of $\mathrm{SO}(d,d)$.

References

[1] C.M. Hull, *A geometry for non-geometric string backgrounds*, *JHEP* **10** (2005) 065 [hep-th/0406102].

[2] B.R. Greene, A.D. Shapere, C. Vafa and S.-T. Yau, *Stringy cosmic strings and noncompact Calabi-Yau manifolds*, *Nucl. Phys.* **B 337** (1990) 1.

[3] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [hep-th/9602022].

[4] A. Kumar and C. Vafa, *U-manifolds*, *Phys. Lett.* **B 396** (1997) 85 [hep-th/9611007].

[5] S. Hellerman, J. McGreevy and B. Williams, *Geometric constructions of nongeometric string theories*, *JHEP* **01** (2004) 024 [hep-th/0208174].

[6] A. Dabholkar and C. Hull, *Duality twists, orbifolds and fluxes*, *JHEP* **09** (2003) 054 [hep-th/0210209].

[7] S. Kachru, M.B. Schulz, P.K. Tripathy and S.P. Trivedi, *New supersymmetric string compactifications*, *JHEP* **03** (2003) 061 [hep-th/0211182].

[8] C.M. Hull and A. Catal-Ozer, *Compactifications with S-duality twists*, *JHEP* **10** (2003) 034 [hep-th/0308133].

[9] A. Flournoy, B. Wecht and B. Williams, *Constructing nongeometric vacua in string theory*, *Nucl. Phys.* **B 706** (2005) 127 [hep-th/0404217].

[10] U. Lindström, R. Minasian, A. Tomasiello and M. Zabzine, *Generalized complex manifolds and supersymmetry*, *Commun. Math. Phys.* **257** (2005) 235 [hep-th/0405085].

[11] C.M. Hull and R.A. Reid-Edwards, *Flux compactifications of string theory on twisted tori*, hep-th/0503114.

[12] A. Dabholkar and C. Hull, *Generalised T-duality and non-geometric backgrounds*, hep-th/0512005.

[13] J. Shelton, W. Taylor and B. Wecht, *Nongeometric flux compactifications*, *JHEP* **10** (2005) 085 [hep-th/0508133].

[14] A. Lawrence, M.B. Schulz and B. Wecht, *D-branes in nongeometric backgrounds*, hep-th/0602025.

[15] M.B. Schulz, *Superstring orientifolds with torsion: O5 orientifolds of torus fibrations and their massless spectra*, *Fortschr. Phys.* **52** (2004) 963 [hep-th/0406001].

[16] A. Flournoy and B. Williams, *Nongeometry, duality twists and the worldsheet*, *JHEP* **01** (2006) 166 [hep-th/0511126].

[17] K.S. Narain, M.H. Sarmadi and C. Vafa, *Asymmetric orbifolds*, *Nucl. Phys.* **B 288** (1987) 551.

[18] This can be shown using the results of: E. Witten, *Non-perturbative superpotentials in string theory*, *Nucl. Phys.* **B 474** (1996) 343 [hep-th/9604030].

[19] J. Scherk and J.H. Schwarz, *How to get masses from extra dimensions*, *Nucl. Phys.* **B 153** (1979) 61.

[20] J. Scherk and J.H. Schwarz, *Spontaneous breaking of supersymmetry through dimensional reduction*, *Phys. Lett.* **B 82** (1979) 60.

[21] A. Salam and E. Sezgin, *D = 8 supergravity*, *Nucl. Phys.* **B 258** (1985) 284.

[22] E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, *Duality of type-II 7-branes and 8-branes*, *Nucl. Phys.* **B 470** (1996) 113 [hep-th/9601150].

[23] I.V. Lavrinenko, H. Lu and C.N. Pope, *Fibre bundles and generalised dimensional reductions*, *Class. and Quant. Grav.* **15** (1998) 2239 [hep-th/9710243].

[24] N. Kaloper, R.R. Khuri and R.C. Myers, *On generalized axion reductions*, *Phys. Lett.* **B 428** (1998) 297 [hep-th/9803066].

[25] C.M. Hull, *Massive string theories from M-theory and F-theory*, *JHEP* **11** (1998) 027 [hep-th/9811021].

[26] C.M. Hull, *Gauged D = 9 supergravities and Scherk-Schwarz reduction*, *Class. and Quant. Grav.* **21** (2004) 509 [hep-th/0203146].

[27] E. Bergshoeff, M. de Roo and E. Eyras, *Gauged supergravity from dimensional reduction*, *Phys. Lett.* **B 413** (1997) 70 [hep-th/9707130].

[28] P. Meessen and T. Ortin, *An* SL(2, $\mathbb{Z}$) *multiplet of nine-dimensional type-II supergravity theories*, *Nucl. Phys.* **B 541** (1999) 195 [`hep-th/9806120`].

[29] P.M. Cowdall, H. Lu, C.N. Pope, K.S. Stelle and P.K. Townsend, *Domain walls in massive supergravities*, *Nucl. Phys.* **B 486** (1997) 49 [`hep-th/9608173`].

[30] P.M. Cowdall, *Novel domain wall and minkowski vacua of* $D = 9$ *maximal SO(2) gauged supergravity*, `hep-th/0009016`.

[31] E. Bergshoeff, U. Gran and D. Roest, *Type iib seven-brane solutions from nine-dimensional domain walls*, *Class. and Quant. Grav.* **19** (2002) 4207 [`hep-th/0203202`].

[32] N. Kaloper and R.C. Myers, *The O(dd) story of massive supergravity*, *JHEP* **05** (1999) 010 [`hep-th/9901045`].

[33] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *Consistent group and coset reductions of the bosonic string*, *Class. and Quant. Grav.* **20** (2003) 5161 [`hep-th/0306043`].

[34] E. Bergshoeff et al., *The bianchi classification of maximal* $D = 8$ *gauged supergravities*, *Class. and Quant. Grav.* **20** (2003) 3997 [`hep-th/0306179`].

[35] N. Alonso Alberca et al., *Domain walls of* $D = 8$ *gauged supergravities and their* $D = 11$ *origin*, *JHEP* **06** (2003) 038 [`hep-th/0303113`].

[36] B. de Wit, H. Samtleben and M. Trigiante, *On lagrangians and gaugings of maximal supergravities*, *Nucl. Phys.* **B 655** (2003) 93 [`hep-th/0212239`].

[37] S.F. Hassan, *T-duality, space-time spinors and RR fields in curved backgrounds*, *Nucl. Phys.* **B 568** (2000) 145 [`hep-th/9907152`].

[38] S.F. Hassan, *SO(d,d) transformations of Ramond-Ramond fields and space-time spinors*, *Nucl. Phys.* **B 583** (2000) 431 [`hep-th/9912236`].

[39] S.F. Hassan, *Supersymmetry and the systematics of T-duality rotations in type-II superstring theories*, *Nucl. Phys.* **102** *(Proc. Suppl.)* (2001) 77 [`hep-th/0103149`].

[40] J.P. Gauntlett, N.W. Kim, D. Martelli and D. Waldram, *Fivebranes wrapped on SLAG three-cycles and related geometry*, *JHEP* **11** (2001) 018 [`hep-th/0110034`].

[41] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G-structures and wrapped NS5-branes*, *Commun. Math. Phys.* **247** (2004) 421 [`hep-th/0205050`].

[42] T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, `math.dg/0102142`.

[43] T. Friedrich and S. Ivanov, *Killing spinor equations in dimension 7 and geometry of integrable* G_2-*manifolds*, `math.dg/0112201`.

[44] S. Ivanov, *Connection with torsion, parallel spinors and geometry of Spin(7) manifolds*, `math.dg/0111216`.

[45] S. Gurrieri, J. Louis, A. Micu and D. Waldram, *Mirror symmetry in generalized Calabi-Yau compactifications*, *Nucl. Phys.* **B 654** (2003) 61 [`hep-th/0211102`].

[46] J.P. Gauntlett and S. Pakis, *The geometry of* $D = 11$ *Killing spinors*, *JHEP* **04** (2003) 039 [`hep-th/0212008`].

[47] P. Kaste, R. Minasian, M. Petrini and A. Tomasiello, *Kaluza-Klein bundles and manifolds of exceptional holonomy*, *JHEP* **09** (2002) 033 [`hep-th/0206213`].

[48] S. Gurrieri and A. Micu, *Type IIB theory on half-flat manifolds*, *Class. and Quant. Grav.* **20** (2003) 2181 [`hep-th/0212278`].

[49] P. Kaste, R. Minasian, M. Petrini and A. Tomasiello, *Nontrivial RR two-form field strength and* SU(3)-*structure*, *Fortschr. Phys.* **51** (2003) 764 [`hep-th/0301063`].

[50] K. Behrndt and C. Jeschek, *Fluxes in M-theory on 7-manifolds and G structures*, *JHEP* **04** (2003) 002 [`hep-th/0302047`].

[51] J.P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, *Phys. Rev.* **D 69** (2004) 086002 [`hep-th/0302158`].

[52] J.P. Gauntlett, J.B. Gutowski and S. Pakis, *The geometry of D = 11 null Killing spinors*, *JHEP* **12** (2003) 049 [`hep-th/0311112`].

[53] K. Behrndt and C. Jeschek, *Fluxes in M-theory on 7-manifolds: G-structures and superpotential*, *Nucl. Phys.* **B 694** (2004) 99 [`hep-th/0311119`].

[54] S. Fidanza, R. Minasian and A. Tomasiello, *Mirror symmetric* SU(3)-*structure manifolds with NS fluxes*, *Commun. Math. Phys.* **254** (2005) 401 [`hep-th/0311122`].

[55] A.R. Frey, *Notes on* SU(3) *structures in type-IIB supergravity*, *JHEP* **06** (2004) 027 [`hep-th/0404107`].

[56] C. Jeschek, *Generalized Calabi-Yau structures and mirror symmetry*, `hep-th/0406046`.

[57] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, *Supersymmetric backgrounds from generalized Calabi-Yau manifolds*, *JHEP* **08** (2004) 046 [`hep-th/0406137`].

[58] S. Gurrieri, A. Lukas and A. Micu, *Heterotic on half-flat*, *Phys. Rev.* **D 70** (2004) 126009 [`hep-th/0408121`].

[59] T. House and A. Micu, *M-theory compactifications on manifolds with G_2 structure*, *Class. and Quant. Grav.* **22** (2005) 1709 [`hep-th/0412006`].

[60] C. Jeschek and F. Witt, *Generalised G_2-structures and type-IIB superstrings*, *JHEP* **03** (2005) 053 [`hep-th/0412280`].

[61] A. Tomasiello, *Topological mirror symmetry with fluxes*, *JHEP* **06** (2005) 067 [`hep-th/0502148`].

[62] K. Behrndt, M. Cvetič and P. Gao, *General type-IIB fluxes with* SU(3) *structures*, *Nucl. Phys.* **B 721** (2005) 287 [`hep-th/0502154`].

[63] T. House and E. Palti, *Effective action of (massive) IIA on manifolds with* SU(3) *structure*, *Phys. Rev.* **D 72** (2005) 026004 [`hep-th/0505177`].

[64] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, *Generalized structures of N = 1 vacua*, *JHEP* **11** (2005) 020 [`hep-th/0505212`].

[65] M. Grana, J. Louis and D. Waldram, *Hitchin functionals in N = 2 supergravity*, *JHEP* **01** (2006) 008 [`hep-th/0505264`].

[66] M. Haack, J. Louis and H. Singh, *Massive type-IIA theory on K3*, *JHEP* **04** (2001) 040 [`hep-th/0102110`].

[67] H. Singh, *Romans type-IIA theory and the heterotic strings*, *Phys. Lett.* **B 545** (2002) 403 [`hep-th/0201206`].

[68] H. Singh, *Duality symmetric massive type-II theories in $D = 8$ and $D = 6$*, *JHEP* **04** (2002) 017 [`hep-th/0109147`].

[69] G. Lopes Cardoso et al., *Non-Kähler string backgrounds and their five torsion classes*, *Nucl. Phys.* **B 652** (2003) 5 [`hep-th/0211118`].

[70] G. Lopes Cardoso, G. Curio, G. Dall'Agata and D. Lüst, *Heterotic string theory on non-Kähler manifolds with H-flux and gaugino condensate*, *Fortschr. Phys.* **52** (2004) 483 [`hep-th/0310021`].

[71] G. Dall'Agata and N. Prezas, *$N = 1$ geometries for M-theory and type-IIA strings with fluxes*, *Phys. Rev.* **D 69** (2004) 066004 [`hep-th/0311146`].

[72] G. Dall'Agata, *On supersymmetric solutions of type-IIB supergravity with general fluxes*, *Nucl. Phys.* **B 695** (2004) 243 [`hep-th/0403220`].

[73] G. Lopes Cardoso, G. Curio, G. Dall'Agata and D. Lüst, *Gaugino condensation and generation of supersymmetric 3-form flux*, *JHEP* **09** (2004) 059 [`hep-th/0406118`].

[74] J. Maharana and J.H. Schwarz, *Noncompact symmetries in string theory*, *Nucl. Phys.* **B 390** (1993) 3 [`hep-th/9207016`].

[75] S.F. Hassan, *O(d,d:R) deformations of complex structures and extended world sheet supersymmetry*, *Nucl. Phys.* **B 454** (1995) 86 [`hep-th/9408060`].